Does light exert Abraham's force in a transparent medium? * ‡

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Abstract: Abraham's force seems to have been observed in very low frequency experiments, but the existence of an Abraham's force exerted by light is still to be proved. In fact, the only experiments performed with light have measured the radiation pressure exerted at the interface between different dielectric media, and their value for inferring conclusions about the existence of Abraham's force is at best opinable. Twenty years ago Brevik proposed an experiment for checking what forces light exert in the interior of a transparent medium. It is argued that his proposal is feasible today with a detection method now at hand, thanks to the great improvements achieved in manufacturing optical fibres with the required physical properties.

1. - Introduction

Let us figure out a physicist who has engaged in even a cursory survey of the huge amount of theoretical literature accumulated in more than one century about the forces that electromagnetism supposedly produces in dielectric media: he has patiently compared the divergent outcomes that eminent theoreticians have arrived at; he has doubtfully pondered the relative advantages of the microscopic versus the macroscopic approach; he has followed the alternating fortunes of electrostriction and magnetostriction [1)]; he has wondered in disbelief why the supposedly universal tool of quantum mechanics has been so inept, in a time span of seventy years, in shedding some light on this issue. While challenged by such a confusing state of affairs, he has even been summoned to surrender, and recognize the evident vanity of the whole effort [2)].

No wonder whether this physicist, like the simple Simon of the stinging tale [11] written by J.L. Synge and entitled *"On the present status of the electromagnetic energy tensor"*, might feel that his very perception of an objective reality is at stake, and that, as an antidote, he had better giving a look to whatever relevant information experimentalists may have gathered on these forces, in the hope that the latter have turned out to behave like classical, macroscopic forces usually do, and that they have not been found to depend on the mental act of splitting the overall energy tensor in two.

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1) Accounted for already in the seminal works by Helmholtz [1],[2], they have no place in the relativistic proposals set forth by Minkowski [3] and Abraham [4], [5], but are present (with a different shape) e.g. in the paper by Einstein and Laub [6].

2) For instance, in Refs. [7]-[10] it is argued, under natural assumptions like those introduced in Ref. [9], that the way one splits the overall macroscopic energy tensor into a matter term and a radiation term is just a convention, merely dictated by reasons of convenience, and maybe of elegance.

2. The experimental evidence for Abraham's force

Given the huge amount of theoretical literature, our natural philosopher might expect to be confronted with an equally fair production on the experimental side, but to his dismay, he will discover that all the papers dealing with experimental findings can be easily contained in a small drawer [3].

Clear evidence for the Abraham force [4] was retrieved with two kinds of experiments, both availing of low frequency electromagnetic fields [15]-[17]. One experiment performed by Walker and Lahoz will be summarized here, since it seems [12] to have provided the most unambiguous outcome up to date.

To do so, let us first specialize the expression of Abraham's generalized force density given in the Appendix for the case when a global coordinate system exist in which the components of the metric are:

$$g_{ik} = \eta_{ik} \equiv diag(1, 1, 1, -1), \tag{1}$$

and the components of the four-velocity of matter are

$$u^1 = u^2 = u^3 = 0, \; u^4 = 1. \tag{2}$$

Let us assume also the vanishing of the four-current $\mathbf{s}^i$; then eq. (A33) reduces to:

$$\mathbf{f}_\rho = -\frac{\epsilon\mu - 1}{\mu} \left[F_{4\lambda} F_\rho{}^\lambda\right]_{,4}, \quad \mathbf{f}_4 = 0, \tag{3}$$

where ϵ and μ stand respectively for the dielectric constant and for the permeability of a homogeneous and isotropic medium at rest. Greek indices run from 1 to 3 and label the spatial components, while a comma signals ordinary differentiation. Then in m.k.s. units Abraham's ordinary force density takes the well known expression:

$$\vec{\mathbf{f}} = \frac{n^2 - 1}{c^2} \frac{\partial}{\partial t} \left(\vec{E} \wedge \vec{H}\right), \tag{4}$$

where n is the refractive index of the medium, and c is the velocity of light *in vacuo*.

In the experiment [16] of Walker and Lahoz a disk of barium titanate, whose relative dielectric constant was near to 4000, had a small cylindrical hole pierced at its center, and both the inner and the outer cylindrical surfaces were coated with metallic films. By sending charges on these electrodes a radial electric field could be set up within the disk. The latter was suspended to a tungsten wire, and the torsional pendulum obtained

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[3] See, e.g., the accurate report [12] published by Brevik in 1979; to our knowledge, just one further finding related to our issue, about the angular momentum of radiation in dielectrics [13], has occurred in the meantime.

[4] A simple derivation of Abraham's energy tensor, found by W. Gordon [14] in 1923, is outlined in the Appendix.

in this way was subjected to a uniform magnetic field of constant value, parallel to the suspending wire. When the radial electric field was allowed to vary according to a sine law at the resonance frequency of the pendulum, steady oscillations were observed to set in, and their amplitude was measured with an optical lever. From the known parameters of the experimental device a value for $\vec{\mathbf{f}}_{exp.}$ was inferred, in good agreement with the theoretical prediction of eq. [4], as far as can be achieved in a delicate experiment of this sort.

3. Reviving a proposal by Brevik

It would be interesting to learn whether light exerts Abraham's force on a transparent medium crossed by it. Since in eq. (4) the product $\vec{E} \wedge \vec{H}$ is differentiated with respect to time, a large value of the instantaneous force density is predicted in this case, but the direct observation of a mechanical force at the frequency of light is beyond reach also for present day experimenters. Brevik [18] made some proposals for detecting the Abraham's force of light in an indirect way, and one of them will be reconsidered here. An order of magnitude estimate of several quantities intervening in the experiment is needed for assessing whether the latter is actually feasible.

Imagine a single-mode optical fibre [5)], whose length is $L = 1km$. The mathematical description of the eigenmodes of the electromagnetic field propagating in such a fibre is rather involved [21], but if we content ourselves with an order of magnitude evaluation, the actual field for the so called transverse electric mode can be replaced with the field of a plane wave propagating along the axis of the fibre, whose intensity is constant within the fibre core, and vanishes abruptly at the core radius. Let the axis of the fibre coincide with the x axis of a Cartesian co-ordinate system, and the electromagnetic plane wave be a linearly polarized one[6)] with the nonvanishing components:

$$E_y = E_0 cos(k_1 x - \omega_1 t) cos(k_2 x - \omega_2 t), \tag{5}$$

$$B_z = \sqrt{\epsilon_0 \epsilon \mu_0 \mu} E_0 cos(k_1 x - \omega_1 t) cos(k_2 x - \omega_2 t), \tag{6}$$

where $\omega_1 \ll \omega_2$, and

$$\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = \frac{c}{n}, \tag{7}$$

[5)] For light with a wavelength in vacuo $\lambda = 1.3\mu m$, as will be chosen here, a single-mode fibre is typically constituted by a cylindrical doped glass core with a radius $r_1 = 5\mu m$, surrounded by a cylindrical shell of pure silica glass with a slightly lower refractive index of radius $r_2 = 60\mu m$, and by a further cylindrical coating made of acrylic resin of still lower refractivity, up to an outer radius $r_3 = 125\mu m$. For an account of the way optical fibres behave see e.g. Refs. [19] and [20].

[6)] Polarization preserving optical fibres are a recent achievement; in usual fibres the polarization is elliptical with irregular variations of the parameters of the ellipse as one proceeds along the axis. This circumstance will not affect our order of magnitude estimate.

i.e. we imagine sending[7)] into the fibre linearly polarized light whose amplitude is modulated with the low angular frequency ω_1. Let us set $\mu = 1$, as appropriate for glass. Then Abraham's force density, defined by eq. (4), is directed along the axis of the fibre and takes the value:

$$\mathbf{f} = n\frac{n^2-1}{4c}\epsilon_0 E_0^2 \frac{\partial}{\partial t}\left\{[1+cos2(k_1x-\omega_1 t)]\,[1+cos2(k_2x-\omega_2 t)]\right\}. \tag{8}$$

The right-hand side of eq. (8) contains a nonvanishing component at low frequency that is potentially observable; it reads:

$$\mathbf{f}_{low} = n\frac{n^2-1}{2c}\epsilon_0 E_0^2 \omega_1 sin2(k_1x-\omega_1 t). \tag{9}$$

The average intensity of the radiation described by eqs. (5) and (6) is $\bar{I} = \frac{1}{4}\epsilon_0 c E_0^2$. Let us assume for E_0 the value

$$E_0 = 10^6 \frac{volt}{m};$$

then the previously quoted size of the core radius allows the fibre to carry the power of $\sim 60mW$, an acceptable choice. Imagine now that the fibre is wound, together with a second one, on a cylinder with radius $R = 10cm$; this move looks possible without altering in an appreciable way the fields prevailing in the core. The cylinder is suspended to such a tungsten wire as to constitute a torsion pendulum whose resonance angular frequency ω is

$$\omega = 2\omega_1 = 2\pi s^{-1}.$$

Due to the low frequency component of Abraham's force the first fibre will exert on the torsion pendulum the torque[8)], say:

$$M = \pi r_1^2 L \mathbf{f}_{low} R = M_0 sin\omega t, \tag{10}$$

where, due to the previous choices, and since in the core $n = 1.44$ at the wavelength chosen for the light:

$$M_0 = 0.63 \times 10^{-15} Nm.$$

The deflection angle ϑ of the torsion pendulum shall fulfil the equation

$$I\ddot{\vartheta} + d\dot{\vartheta} + k\vartheta = M_0 sin\omega t, \tag{11}$$

[7)] A fibre laser [20] can be directly joined to the fibre without interposition of reflecting surfaces. Eqs. (5) and (6) do not account for losses; this is a reasonable approximation, since the attenuation of the fibre considered above can be as low as 0.3 dB, the limiting value due to Rayleigh scattering.

[8)] If both ends of the fibre are left loose from the torsion pendulum and rigidly clamped, surface forces occurring at the extremities will not contribute to the torque.

where I, d and k are respectively the momentum of inertia, the damping parameter and the elastic constant. For very small damping the resonance angular frequency ω is practically equal to the natural frequency of the undamped pendulum, *i.e.* $\omega = \sqrt{k/I}$. The standing oscillation at that frequency, according to eq. (11), obeys the law:

$$\vartheta = -\frac{M_0}{\omega d} cos\omega t. \tag{12}$$

We can estimate the damping parameter d by reminding that the time constant τ of the damped oscillations is related to d by the equation $\tau = 2I/d$. Since a reasonable value for the time constant is $\tau \approx 10^3 s$, while $I \approx 2.5 \times 10^{-3} kg \times m^2$, the standing oscillation at resonance will occur with the tangential velocity:

$$v = R\dot{\vartheta} = \frac{M_0 R}{d} sin\omega t,$$

where one can set

$$\frac{M_0 R}{d} \approx 1.26 \times 10^{-11} ms^{-1}.$$

We propose availing of the second fibre to detect this velocity. We shall disregard the fact that the fibre, whose length can be again $L = 1km$, is wound on a cylinder, and reason as if it had been uncoiled. We shall assume that when this fibre is at rest its core, parallel to, say, the x axis, is travelled by the unmodulated, linearly polarized light [9] whose components, written *more relativistico*, are:

$$F_{24} = E'_0 cos(k_2 x - \omega_2 t), \tag{13}$$

$$F_{12} = \sqrt{\epsilon\mu} E'_0 cos(k_2 x - \omega_2 t). \tag{14}$$

We imagine however that the velocity field has followed the fibre in the deformation due to the unwinding action; given the smallness of v, we can approximate the four-velocity of the uncoiled fibre as:

$$u^i \approx (\beta, 0, 0, 1), \tag{15}$$

where

$$\beta = \frac{M_0 R}{cd} sin\omega t \approx -\frac{M_0 R}{cd} sin[2(k_1 x - \omega_1 t)], \tag{16}$$

because $k_1 x$ is a negligible term also for $x = L$. Since the fibre is subjected to this velocity field, eqs. (13) and (14) no longer provide an exact solution to the Maxwell's equations: the constitutive relation has changed, however slightly, in keeping with eq. (A14). Due to the smallness of the change, calculating a first order perturbation may suffice. If δF^{rs} is

[9] For reasons apparent in the sequel, the second fibre needs to be polarization preserving.

the change undergone by the unperturbed F^{rs}, the corresponding change in H^{rs} will be given by:

$$\delta H^{rs} = \frac{1}{\mu}\delta F^{rs} + \frac{\epsilon\mu - 1}{\mu}(\delta_4^r \delta F^{4s} - \delta_4^s \delta F^{4r}) + \frac{\epsilon\mu - 1}{\mu}\beta(\delta_1^r F^{4s} - \delta_1^s F^{4r} + \delta_4^r F^{s1} - \delta_4^s F^{r1}). \quad (17)$$

Hence in the present case the first order correction to one unperturbed set of Maxwell's equations reads:

$$\frac{1}{\mu}\delta F^{\lambda\rho}_{,\rho} + \epsilon\delta F^{\lambda 4}_{,4} = \frac{\epsilon\mu - 1}{\mu}\left[(\beta F^{4\lambda})_{,1} - (\beta F^{1\lambda})_{,4})\right] \quad (18)$$

for $r = \lambda$, and

$$\epsilon\delta F^{4\rho}_{,\rho} = 0, \quad (19)$$

for $r = 4$, while the correction to the other unperturbed set is:

$$\delta F_{[ik,m]} = 0. \quad (20)$$

Eqs. (18)-(20) display in δF^{ik} the same form as the unperturbed equations do in F^{ik}, with the exception of the component with $\lambda = 2$ of eq. (18), that reads:

$$\delta F^{21}_{,1} + \epsilon\mu\delta F^{24}_{,4} = 2(\epsilon\mu - 1)(\beta F^{42})_{,1} \quad (21)$$

since in our case:

$$(\beta F^{42})_{,1} = -(\beta F^{12})_{,4}.$$

Therefore only δF^{21} and δF^{24} need to be nonvanishing. Let us set:

$$\zeta = (2k_1 + k_2)x - (2\omega_1 + \omega_2)t,$$
$$\eta = (2k_1 - k_2)x - (2\omega_1 - \omega_2)t,$$

and

$$C = \tfrac{1}{2}(\epsilon\mu - 1)\frac{M_0 R E_0'}{cd};$$

then a physically appropriate particular solution is given by:

$$\delta F_{12} = C\left\{sin\zeta + sin\eta + (2k_1 + k_2)xcos\zeta + (2k_1 - k_2)xcos\eta\right\}, \quad (22)$$

$$\delta F_{42} = -\frac{C}{\sqrt{\epsilon\mu}}\left\{(2k_1 + k_2)xcos\zeta + (2k_1 - k_2)xcos\eta\right\}. \quad (23)$$

We are interested in the behaviour of δF_{ik} near the end of the second fibre, therefore we can retain only the terms that grow linearly with x. Since $k_1 \ll k_2$, we eventually write:

$$\delta F_{12} \approx 2\pi(\epsilon\mu - 1)\frac{M_0 R E_0'}{cd}\frac{x}{\lambda_2} sin\omega t\; sin(k_2 x - \omega_2 t), \tag{24}$$

$$\delta F_{24} \approx 2\pi\frac{\epsilon\mu - 1}{\sqrt{\epsilon\mu}}\frac{M_0 R E_0'}{cd}\frac{x}{\lambda_2} sin\omega t\; sin(k_2 x - \omega_2 t), \tag{25}$$

where λ_2 is the wavelength of light *in vitro*. We remark that δF_{ik}, at variance with the unperturbed F_{ik}, exhibits a "sin" rather than a "cos" dependence on the argument $k_2 x - \omega_2 t$. The detection of the signal could be done by adding [10)] at the end of the second fibre the field with the nonvanishing components:

$$F_{12} = \sqrt{\epsilon\mu} E_0' sin(k_2 x - \omega_2 t). \tag{26}$$

$$F_{24} = E_0' sin(k_2 x - \omega_2 t). \tag{27}$$

Then the square of the electric field E_y has a component whose low frequency average, that can be perceived by a photodetector, reads:

$$(E_y^2)_{av.} = (E_0')^2[1 - 2. \times 10^{-10} sin\omega t]. \tag{28}$$

Although quite small, when compared with the background, the signal caused by the component of $(E_y^2)_{av.}$ at the resonance frequency of the torsion pendulum seems within reach of the present day techniques of digital phase-sensitive detection. Its observation would provide evidence that Abraham's force is indeed exerted by light inside a dielectric; in fact, electrostriction [7], [12] in glass should produce a signal that is smaller in magnitude, and opposite in sign with respect to the one calculated above.

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[10)] Fibre couplers just built with this scope in mind are commonly available. Of course the whole procedure of detection is only possible if the second fibre is polarization preserving, and if the coherence length of the light sent in the second fibre is longer than the fibre itself. The latter condition ultimately means [20] a request about the phase and the amplitude stability of the laser source that can be met with by present day lasers.

Appendix: Gordon's derivation of Abraham's tensor

The general relativistic argument conceived by W. Gordon has been retrieved in the limbo of the forgotten papers [22]. It runs as follows. In a four-dimensional differentiable manifold let us consider a contravariant skew tensor density $\mathbf{H}^{ik}$, a covariant skew tensor F_{ik}, and write the naturally invariant equations:

$$\mathbf{H}^{ik}_{,k} = \mathbf{s}^i, \tag{A1}$$

$$F_{[ik,m]} = 0. \tag{A2}$$

The vector density $\mathbf{s}^i$, defined by the left-hand side of eq. (A1), represents the electric four-current density, while the comma signals ordinary differentiation, and we have set $F_{[ik,m]} \equiv \frac{1}{3}(F_{ik,m} + F_{km,i} + F_{mi,k})$. Equations (A1) and (A2) express Maxwell's equations in general curvilinear co-ordinates; they need to be complemented with the constitutive relation of electromagnetism, a tensor equation that uniquely defines *e.g.* $\mathbf{H}^{ik}$ in terms of F_{ik} and of whatever additional fields may be needed for specifying the properties of the electromagnetic medium. When the assumed dependence of $\mathbf{H}^{ik}$ on F_{ik} is algebraic and linear, the constitutive relation reads:

$$\mathbf{H}^{ik} = \tfrac{1}{2}\mathbf{X}^{ikmn} F_{mn}, \tag{A3}$$

and the electromagnetic properties of the medium are summarized by the four-index tensor density $\mathbf{X}^{ikmn}$. In the case of vacuum one writes:

$$\mathbf{X}^{ikmn}_{vac.} = \sqrt{g}(g^{im}g^{kn} - g^{in}g^{km}); \tag{A4}$$

the constitutive relation entails only the metric tensor g_{ik} and $g \equiv -det(g_{ik})$ in the way known from general relativity. In this case Maxwell's equations are usually written as:

$$\mathbf{F}^{ik}_{,k} = \mathbf{s}^i, \tag{A5}$$

$$F_{[ik,m]} = 0, \tag{A6}$$

in terms of a skew tensor F_{ik} and of the contravariant tensor density $\mathbf{F}^{ik} \equiv \sqrt{g}g^{im}g^{kn}F_{mn}$. A general skew tensor F_{ik} can always be written as the sum of the curl of a potential and of the dual to the curl of an "antipotential":

$$F_{ik} = \varphi_{k,i} - \varphi_{i,k} + e_{ik}{}^{mn}(\psi_{n,m} - \psi_{m,n}), \tag{A7}$$

where $e_{ik}{}^{mn}$ is the tensor obtained from the Ricci-Levi Civita symbol $\mathbf{e}^{ikmn}$ in the usual way. Starting from the Lagrangian density

$$\mathbf{L} = \tfrac{1}{4}\mathbf{F}^{ik}F_{ik} - \mathbf{s}^i\varphi_i \tag{A8}$$

for the general relativistic vacuum, both sets of Maxwell's equations can be derived through the Hamilton principle, by asking that the variations of $A = \int \mathbf{L}\, dS$ ($dS = dx^1\, dx^2\, dx^3\, dx^4$)

with respect to φ_i and to ψ_i separately vanish [23]. The well known form of the energy tensor density $\mathbf{T}_{ik}$ for the electromagnetic field *in vacuo* is obtained through the method [24] inaugurated by Hilbert, *i.e.* by carrying out the Hamiltonian derivative of the Lagrangian density (A8) with respect to g^{ik}:

$$\mathbf{T}_{ik} \equiv 2\frac{\delta \mathbf{L}}{\delta g^{ik}} = \mathbf{F}_i{}^n F_{kn} - \tfrac{1}{4} g_{ik} \mathbf{F}^{mn} F_{mn}. \tag{A9}$$

The derivation of the energy tensor must be performed by keeping into account the homogeneous field equations obtained above; they dictate that F_{ik} is the curl of a four-vector φ_i and does not contain the metric.

The constitutive relation for a linear, nondispersive medium will have in general the form of eq. (A3). Gordon deals with a medium that in addition is homogeneous and isotropic, when considered at rest; its constitutive relation can be written in a simple form, due to Minkowski [3], that can be extended without change to general relativity. Let

$$u^i = \frac{dx^i}{\sqrt{-ds^2}} \tag{A10}$$

be the four-velocity of matter, for which $u_i u^i = -1$. One defines the four-vectors

$$F_i = F_{ik} u^k, \quad H_i = H_{ik} u^k, \tag{A11}$$

where $H_{ik} \equiv (1/\sqrt{g}) g_{ip} g_{kq} \mathbf{H}^{pq}$ is the covariant tensor associated with $\mathbf{H}^{ik}$. Then the above mentioned constitutive relation simply reads:

$$H_i = \epsilon F_i, \tag{A12}$$

$$u_i F_{km} + u_k F_{mi} + u_m F_{ik} = \mu \big(u_i H_{km} + u_k H_{mi} + u_m H_{ik} \big), \tag{A13}$$

where ϵ and μ are the dielectric constant and the permeability, and do not depend on the chosen event. These eight equations, that entail two identities, are equivalent [25] to the six equations:

$$\mu H^{ik} = F^{ik} + (\epsilon\mu - 1)(u^i F^k - u^k F^i). \tag{A14}$$

Since the right-hand side of eq. (A14) can be rewritten as

$$F_{rs} \left\{ g^{ir} g^{ks} - (\epsilon\mu - 1) \left(u^i u^r g^{ks} + u^k u^s g^{ir} \right) \right\},$$

and, due to the antisymmetry of F_{rs}, one can freely add the term $(\epsilon\mu - 1)^2 u^i u^k u^r u^s$ within the curly brackets, the constitutive relation eventually comes to read:

$$\mu H^{ik} = \left[g^{ir} - (\epsilon\mu - 1) u^i u^r \right] \left[g^{ks} - (\epsilon\mu - 1) u^k u^s \right] F_{rs}. \tag{A15}$$

The "effective metric tensor":

$$\gamma^{ik} = g^{ik} - (\epsilon\mu - 1) u^i u^k, \tag{A16}$$

whose inverse is

$$\gamma_{ik} = g_{ik} + \big(1 - \frac{1}{\epsilon\mu}\big)u_i u_k, \tag{A17}$$

allows one to write eq. (A14) in the form:

$$\mu \mathbf{H}^{ik} = \sqrt{g}\gamma^{ir}\gamma^{ks}F_{rs}. \tag{A18}$$

Since $g \equiv -det(g_{ik})$, we shall pose $\gamma \equiv -det(\gamma_{ik})$. The ratio γ/g is an invariant, and its calculation can be performed in the co-ordinate system in which $u^1 = u^2 = u^3 = 0$; one finds:

$$\gamma = \frac{g}{\epsilon\mu},$$

so that eq. (A15) can be rewritten as

$$\mathbf{H}^{ik} = \sqrt{\frac{\epsilon}{\mu}}\sqrt{\gamma}\gamma^{ir}\gamma^{ks}F_{rs}, \tag{A19}$$

which, *apart from the constant factor* $\sqrt{\epsilon/\mu}$, is just the constitutive relation for the vacuum case in general relativity, when γ_{ik} acts as metric.

We shall henceforth enclose in round brackets the indices which are either moved with γ^{ik} and γ_{ik}, or generated by performing a Hamiltonian derivative with respect to the latter tensors; therefore eq. (A19) will be rewritten as

$$\mathbf{H}^{ik} = \sqrt{\frac{\epsilon}{\mu}}\sqrt{\gamma}F^{(i)(k)}. \tag{A20}$$

Due to the strict analogy with the vacuum case made evident in this way, selecting the form of the Lagrangian density for the electromagnetic field in the medium under question is for Gordon a straightforward matter. He writes:

$$\mathbf{L}' = \tfrac{1}{4}\sqrt{\frac{\epsilon}{\mu}}\sqrt{\gamma}F^{(i)(k)}F_{ik} - \mathbf{s}^i\varphi_i, \tag{A21}$$

where F_{ik} can be defined by:

$$F_{ik} = \varphi_{k,i} - \varphi_{i,k} + \frac{1}{\sqrt{\gamma}}\mathbf{e}_{(i)(k)}{}^{mn}(\psi_{n,m} - \psi_{m,n}). \tag{A22}$$

Equating to zero the independent variations of $\int \mathbf{L}'\,dS$ with respect to φ_i and to ψ_i will produce Maxwell's equations, *a priori* complemented by the constitutive relation (A14).

It is now easy to derive the energy tensor for the electromagnetic field by starting from the derivation that one performs *in vacuo*, when the metric field γ_{ik} is present. In that case Hilbert's procedure leads to write:

$$\delta\mathbf{L} \equiv \tfrac{1}{2}\mathbf{T}_{(i)(k)}\delta\gamma^{ik},$$

hence one gets

$$\mathbf{T}_{(i)}^{\ (k)} = \sqrt{\gamma}\big(F_{ir}F^{(k)(r)} - \tfrac{1}{4}\delta_i^{\ k}F_{rs}F^{(r)(s)}\big). \qquad (A23)$$

For the medium contemplated by Gordon one may be tempted to write:

$$\delta\mathbf{L}' \equiv \tfrac{1}{2}\mathbf{T}'_{(i)(k)}\delta\gamma^{ik}, \qquad (A24)$$

where $\mathbf{L}'$ is given by eq. (A21), and finds

$$\mathbf{T}_{(i)}^{'\ (k)} = F_{ir}\mathbf{H}^{kr} - \tfrac{1}{4}\delta_i^{\ k}F_{rs}\mathbf{H}^{rs}, \qquad (A25)$$

which is just the general relativistic version of the form proposed by Minkowski. Henceforth we shall drop the prime, since its omission will not lead to confusion.

However, $\mathbf{T}_{(i)}^{\ (k)}$ can not be the energy tensor density that we are looking for, because it is defined with respect to the effective metric γ_{ik}, not with respect to the true metric g_{ik}, the only one entitled to account for the structure of space-time and, via the Einstein tensor, for its stress-energy-momentum tensor. In order to find the relation between $\mathbf{T}_{ik}$ and $\mathbf{T}_{(i)(k)}$ we simply need to express $\delta\gamma^{ik}$ in terms of δg^{ik}. From eq. (A10) one obtains the variation of u^i produced by the variation δg^{mn} of the metric:

$$\delta\, u^i = \tfrac{1}{2}u^i u^m u^n \delta g_{mn} = -\tfrac{1}{2}u^i u_m u_n \delta g^{mn},$$

hence from the definition (A16) one gets:

$$\delta\,\gamma^{ik} = \delta\big\{g^{ik} - (\epsilon\mu - 1)u^i u^k\big\} = \delta g^{ik} + (\epsilon\mu - 1)u^i u^k u_m u_n \delta g^{mn}. \qquad (A26)$$

Therefore, since $\mathbf{T}_{ik}\delta g^{ik} = \mathbf{T}_{(i)(k)}\delta\gamma^{ik}$, we find immediately:

$$\mathbf{T}_{\,ik} = \mathbf{T}_{(i)(k)} + (\epsilon\mu - 1)u_i u_k \mathbf{T}_{(m)(n)}u^m u^n. \qquad (A27)$$

The mixed components of $\mathbf{T}_{ik}$ can be obtained by multiplying the left-hand side and the second term at the right-hand side of eq. (A27) by g^{kq}, while the first term at the right-hand side is multiplied by $\gamma^{kq} + (\epsilon\mu - 1)u^k u^q$. Then

$$\mathbf{T}_i^{\ q} = \mathbf{T}_{(i)}^{\ (q)} + (\epsilon\mu - 1)\big\{\mathbf{T}_{(i)(k)}u^k + u_i\mathbf{T}_{(m)(n)}u^m u^n\big\}u^q. \qquad (A28)$$

In a co-ordinate system for which, at a given event, $g_{ik} = \eta_{ik} \equiv diag(1,1,1,-1)$ and $u^1 = u^2 = u^3 = 0,\ u^4 = 1$, the covariant four-vector within the curly brackets has the components $\mathbf{T}_{(\alpha)(4)},\ 0\ (\alpha = 1,2,3)$. But under these circumstances eq. (A27) says that $\mathbf{T}_{(\alpha)(4)} = \mathbf{T}_{\alpha 4}$, hence in a general co-ordinate system one can write:

$$\mathbf{T}_i^{\ q} = \mathbf{T}_{(i)}^{\ (q)} + (\epsilon\mu - 1)\big\{\mathbf{T}_{ik}u^k + u_i\mathbf{T}_{mn}u^m u^n\big\}u^q, \qquad (A29)$$

i.e., according to eq. (A25)

$$T_i^{\ k} = F_{ir}H^{kr} - \tfrac{1}{4}\delta_i^{\ k}F_{rs}H^{rs} - (\epsilon\mu - 1)\Omega_i u^k, \qquad (A30)$$

where Minkowski's "Ruh-Strahl"

$$\Omega^i = -\big(T_n{}^i u^n + u^i T_{mn} u^m u^n\big) \tag{A31}$$

has been introduced. Since $\Omega^i u_i = 0$, by substituting (A30) into (A31) one finds:

$$\Omega^{\,i} = F_m H^{im} - F_m H^m u^i = u_k F_m\big(H^{ik}u^m + H^{km}u^i + H^{mi}u^k\big). \tag{A32}$$

Equations (A30) and (A32) define the extension to general relativity of the energy tensor proposed by Abraham [4], [5] for the electromagnetic field if the medium is homogeneous and isotropic when considered at rest.

The generalized force density due to the electromagnetic field is eventually defined by the covariant divergence:

$$\mathbf{f}_i \equiv -\mathbf{T}_i{}^k{}_{;k}. \tag{A33}$$

References

[1] Helmholtz, H., *Ann. d. Physik u. Chemie*, **13**, 385 (1881).
[2] v. Helmholtz, H., *Ann. d. Physik u. Chemie*, **47**, 1 (1892).
[3] Minkowski, H., Gött. Nachr., Math.-phys. Klasse (1908), 53.
[4] Abraham, M., Rend. Circ. Matem. Palermo **28**, 1 (1909).
[5] Abraham, M., Rend. Circ. Matem. Palermo **30**, 33 (1910).
[6] Einstein, A. and Laub, J., *Ann. d. Physik*, **26**, 541 (1908).
[7] Robinson, F.N.H., *Phys. Reports*, **16**, 313 (1975).
[8] Israel, W., *Gen. Rel. Grav.*, **9**, 451 (1978).
[9] Kranyš, M., *Can. J. Phys.*, **57**, 1022 (1979).
[10] Maugin, G.A., *Can. J. Phys.*, **58**, 1163 (1980).
[11] Synge, J.L., *Hermathena*, **117**, 80 (1974).
[12] Brevik, I., *Phys. Reports*, **52**, 133 (1979).
[13] Kristensen, M. and Woerdman, J.P., *Phys. Rev. Lett*, **72**, 2171 (1994).
[14] Gordon, W., *Ann. d. Physik*, **72**, 421 (1923).
[15] James, R.P., *Proc. Nat. Acad. Sci.*, **61**, 1149 (1968).
[16] Walker, G.B. and Lahoz, D.G., *Nature*, **253**, 339 (1975).
[17] Walker, G.B., Lahoz, D.G. and Walker, G., *Can. J. Phys.*, **53**, 2577 (1975).
[18] Ref. [12], pp. 188-191.
[19] Adams, M.J., *An Introduction to Optical Waveguides*, (J. Wiley) 1981.
[20] Agrawal, G.P., *Nonlinear Fiber Optics*, (Academic Press) 1995.
[21] see e.g. Ref. [19], Ch. 7.
[22] Antoci, S. and Mihich, L., Nuovo Cimento B, **112**, 991 (1997).
[23] Finzi, B., Rend. Accad. Naz. Lincei **12**, 378 (1952).
[24] Hilbert, D., Gött. Nachr., Math.-phys. Klasse (1915), 395.
[25] Nordström, G., Soc. Scient. Fenn., Comm. Phys.-Math. **1.33**, 1 (1923).